FrontierMath Open Problems

Epoch AI

Find an algorithm to determine if a knot has unknotting number one

Introduction

The ability of AI systems to attack mathematical questions is advancing quickly. This article presents a challenge in this direction connected to an open mathematical problem. The problem, stated below, concerns finding an algorithm to determine if a knot has unknotting number equal to one. It is the simplest open case of one of the basic unsolved problems in knot theory, the Unknotting Problem. It is not known at present whether an algorithm exists for either the general case of the Unknotting Problem or for this special case. A solution would be a major achievement in topology.

The Unknotting Number $U(K)$ of a knot K is a classic invariant. It is defined as the smallest number of crossing changes required to change a diagram of K to a diagram of the unknot.

A knot K has unknotting number one if it is nontrivial and there exists some diagram of K in which changing a single crossing turns that diagram into a diagram for the unknot. To establish that K has unknotting number one it suffices to show

1. K is not the unknot
2. There is a diagram of K and a crossing change of that diagram that changes K into the unknot.

Showing that K does not have unknotting number one has been done by computing knot invariants that bound the unknotting number from below. Note that there is an algorithm to decide whether K is unknotted [5] and software to verify this in many cases [3, 2].

Problem Statement

Problem: Give an algorithm that inputs a knot and determines whether the Unknotting Number of the knot is equal to one.

Importance of the Problem

This problem is one of the fundamental questions in low-dimensional topology. It asks a basic question about how hard it is to trivialize a knot. A solution either way would be a major result in Knot Theory.

It also has implications in computational complexity theory. At present we do not know if the problem is undecidable, or if there is a simple algorithm with polynomial running time. Investigations into algorithms related to knots has led to important new insights into the theory of algorithms and computational complexity, going back to work of Dehn in the early 1900's [4].

A third connection is to topological applications in the study of knotted DNA and knotted proteins. A crossing change corresponds to a biological process where enzymes called topoisomerases split and reconnect DNA.

Either a positive or a negative solution to the problem is plausible. Both have been found in related natural topological problems. A positive solution was found for the closely related Unknotting Problem. Haken gave an algorithm to determine whether a knot is trivial (has unknotting number 0) [5]. A negative solution to the existence of an algorithm to a basic topological problem was given by Markov, who showed that there is no algorithm to determine if two smooth 4-manifolds are diffeomorphic [6].

Assessment of the difficulty of the problem

No progress has been made in discovering an algorithm for the Unknotting Problem. A recent result indicates that the general problem of determining Unknotting Number is NP-hard [1], but this is not known for the more limited question of determining if the Unknotting Number is equal to one.

The difficulty in computing the Unknotting Number arises from the need to minimize over all possible diagrams that represent the knot. There is no obvious limit on how complicated a diagram is needed. Examples exists for which the simplest diagram does not exhibit the number of crossings need to unknot.

For the related algorithm to determine whether $U(K) = 0$, i.e whether a knot is the unknot, open source software is available that runs successively on knot diagrams with up to several hundreds of crossings, though not all [3, 2]. A quasi-polynomial running time for a knot triviality algorithm was recently announced by Lackenby.

References

[1] J. Bae. "Diagrammatic Unknotting Number is NP-hard and Computational Link Problems". PhD thesis. Newark: Rutgers University, 2025.

[2] B. A. Burton, R. Budney, W. Pettersson, et al. *Regina: Software for Low-Dimensional Topology*. https://regina-normal.github.io. Software (1999–2025). 2025.

[3] M. Culler et al. *SnapPy, a Computer Program for Studying the Geometry and Topology of 3-Manifolds*. https://snappy.computop.org. Software. 2025.

[4] M. Dehn. "Über die Topologie des dreidimensionalen Raumes". In: *Mathematische Annalen* 69 (1910), pp. 137–168.

[5] W. Haken. "Theorie der Normalflächen". In: *Acta Mathematica* 105 (1961), pp. 245–375. DOI: 10.1007/BF02559591.

[6] A. A. Markov. "Unsolvability of the Homeomorphy Problem". In: *Proceedings of the International Congress of Mathematicians*. Edinburgh: Cambridge University Press, 1960, pp. 300–306.