FrontierMath Open Problems

Epoch AI

Symplectic ball packing

Symplectic ball packing has been studied in many papers, including [4, 5, 3, 1, 2, 6]. In dimension four, it is known that it is possible to fully fill a symplectic ball by k symplectic balls of the same radius whenever $k \geq 10$. Here "fully fill" means that can find a symplectomorphism under which the images of the balls under a symplectomorphism takes up all but ε of the volume of the target ball, for any arbitrarily small $\varepsilon > 0$. However, the proof involved Seiberg–Witten theory and symplectic inflation and is not at all explicit. It is an important open problem to find explicit constructions of these embeddings (among other things, this is related to the famous Nagata conjecture in algebraic curve theory – see e.g. [2]).

Problem difficulty level

While there are explicit constructions of optimal symplectic packings in the case $n = 2$ and $k \leq 9$, no explicit constructions are known for $k \geq 10$. A solution is therefore likely to require fundamental new insights and would likely be considered a significant breakthrough in symplectic geometry, worthy of publication in a top journal. Note that finding a single symplectic embedding (or equivalently Hamiltonian flow) which approximates a full filling with high accuracy is already an important achievement, although this may be possible using a brute force machine learning approach. However, the formulation given below in which a single solution is found for all $\varepsilon > 0$ is significantly harder and is unlikely to have a brute force solution.

Problem formulation: special version

Fix $n = 2$ and $k = 10$. For $i = 1, \ldots, k$, let $B_i \subset \mathbb{R}^{2n}$ denote the closed unit ball centered at $(3i - 3, 0, \ldots, 0)$. For all $\varepsilon > 0$, find an explicit smooth Hamiltonian function $H_\varepsilon : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}$ whose time-1 Hamiltonian flow $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfies $\phi(\sqcup_{i=1}^{k} B_i) \subset B^{2n}(R)$, where $B^{2n}(R) \subset \mathbb{R}^{2n}$ is the closed ball centered at the origin of radius R, and we require the volume ratio of $\bigsqcup_{i=1}^{k} B_i$ to $B^{2n}(R)$ to satisfy

$$\frac{k}{R^{2n}} > 1 - \varepsilon. \tag{1}$$

Problem formulation: general version

For the general version, we ask exactly the same thing, but now for all positive integers $k \geq 10$.

References

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