EPOCH AI

FrontierMath Open Problems

Epoch AI

Finding three integer cubes that sum to 114

Status: open as of July 2026

Problem statement

Goal. Find integers $x, y, z \in \mathbb{Z}$ satisfying

$$x^3 + y^3 + z^3 = 114.$$

A submission is simply an explicit triple (x, y, z). It can be checked by exact integer arithmetic; no proof of minimality is required.

No solution is presently known. Among positive integers k not excluded by the elementary congruence obstruction $k \equiv \pm 4 \pmod 9$, the value 114 is the smallest unresolved case [1]. Permuting the three coordinates gives equivalent solutions.

Background and significance

Every integer cube is congruent to 0 or ± 1 modulo 9. Consequently, the equation

$$x^3 + y^3 + z^3 = k \tag{1}$$

has no integer solution when $k \equiv \pm 4 \pmod 9$. The converse is widely conjectured: Heath-Brown's density heuristic predicts infinitely many solutions for every other k [3, 6]. Since $114 \equiv -3 \pmod 9$, it has no such local obstruction. In fact, any solution must have $x^3 \equiv y^3 \equiv z^3 \equiv -1 \pmod 9$, and hence $x \equiv y \equiv z \equiv -1 \pmod 3$; these restrictions thin the search without making it empty.

The problem also sits near a basic boundary in Diophantine decision problems. Linear and quadratic equations over the integers can be handled algorithmically, while for cubic equations in three or more variables there is currently neither a general decision algorithm nor a proof that no such algorithm can exist [5]. This statement concerns a *family* of input equations. For the single equation above, a positive answer has a finite certificate and is automatically provable in any ordinary formal system strong enough to verify integer arithmetic. Only a negative answer could conceivably be independent of such a system; no independence claim specific to 114 is known. This distinction is useful when invoking Hilbert's tenth problem as motivation [8].

Historically, equation (1) has repeatedly produced solutions whose coordinates are far beyond the ranges suggested by direct search. The cases 33 and 42, for example, were settled only in 2019 and have coordinates larger than 10^{16} [2, 3, 9]. These examples show that failure to find a small solution is weak evidence for nonexistence.

The best current search method

Following Heath-Brown, Lioen and te Riele, and the refinements of Booker and Sutherland, assume after permuting coordinates that $|x| > |y| > |z|$, and put

$$d := |x + y|.$$

The exceptional cases reduce either to $x = -y$ and $z^3 = 114$, or to the Thue equation $x^3 + 2y^3 = 114$, and are readily checked. In the remaining case, the identities behind the factorization of $x^3 + y^3$ show that triples correspond exactly to pairs (d, z) for which

$$z^3 \equiv 114 \pmod{d}, \qquad \Delta(d, z) := 3d(4|114 - z^3| - d^3) \quad \text{is a square.} \tag{2}$$

For $|z| > \sqrt{114}$ one also has

$$0 < d < \alpha|z|, \qquad \alpha = \sqrt[3]{2} - 1 \approx 0.259921. \tag{3}$$

Thus a seemingly three-dimensional search becomes an enumeration of d, followed by a search in the arithmetic progressions containing the cube roots of 114 modulo d. Chinese-remainder enumeration, auxiliary-prime square sieves, and cubic-reciprocity constraints make this practical. For fixed k, the resulting algorithm uses

$$O(B \log\log B \log\log\log B)$$

operations on integers of magnitude $O(B)$ and finds every solution with $\min\{|x|, |y|, |z|\} \leq B$, even when the other two coordinates are much larger [3, 7, 9].

The published computation proceeded in two relevant stages [3]:

- In September 2019, a complete search with $z_{\max} = 10^{17}$ and $d_{\max} = \alpha z_{\max}$ found no solution for 114. Subject to the stated caveat about undetected distributed-computing errors, this rules out every solution with $\min\{|x|, |y|, |z|\} \leq 10^{17}$.
- During 2020, an optimized search used $z_{\max} = 10^{19}$ and $d_{\max} = z_{\max}/54$. It also found no solution for 114. This second rectangle probes likely highly skewed solutions much further, but it is *not* a complete height bound of 10^{19}, because $d_{\max} < \alpha z_{\max}$.

An independent number-field approach represents $x + y$ by norms in $\mathbb{Q}(\sqrt[3]{k})$ and reduces auxiliary searches to integral points on elliptic curves. Grantham and Walsh report that substantial effort with this method for $k = 114$ was likewise unsuccessful [4]. Its failure suggests that the difficulty of 114 is not tied solely to one implementation of the (d, z)-sieve.

Heuristics: why the problem is fair

Let $n_{114}(B)$ count solutions up to permutation with $\max\{|x|, |y|, |z|\} \leq B$. Heath-Brown's heuristic predicts

$$n_{114}(B) \sim \rho_{114} \log B, \qquad \rho_{114} \approx 0.058459. \tag{4}$$

The same leading density is expected for other homogeneous heights, including $\min\{|x|, |y|, |z|\}$ and d. Booker and Sutherland computed the local-density product defining ρ_{114} and compared the analogous prediction, averaged over cube-free $k \leq 1000$, with all known solutions through height 10^{15}; the agreement is strong over more than six thousand sample points [3].

For 114, the leading prediction $e(B) = \rho_{114} \log B$ gives the following scale.

B	10^5	10^8	10^{11}	10^{14}	10^{17}
$e(B)$	0.673	1.077	1.481	1.884	2.288
known solutions	0	0	0	0	0

An empty search is not in serious tension with the conjecture: the prediction is logarithmic, has lower-order terms that are not controlled for an individual k, and does not imply the first solution should occur when $e(B) = 1$.

The directional heuristic in [3, Section 4] permits a more faithful summary of the actual 2020 rectangle. If $R = z_{\max}/d_{\max}$, its predicted solution count has leading form

$$\rho_{114} \log d_{\max} + C - \rho_{114} c \int_R^\infty \frac{K(r)}{r}\,\mathrm{d}r, \quad K(R) = \int_R^\infty \frac{\mathrm{d}r}{\sqrt{4r^3-1}}, \quad c \approx 1.960843. \tag{5}$$

Equating this with the prediction for a complete search gives an "effective complete height"

$$B_{\text{eff}} = \frac{d_{\max}}{\alpha} \exp\left(c \int_{1/\alpha}^{R} \frac{K(r)}{r}\,\mathrm{d}r \right) \approx 3.1 \times 10^{18} \tag{6}$$

for $d_{\max} = 10^{19}/54$ and $R = 54$. This conversion is heuristic, but it incorporates the shape of the searched region rather than treating $z_{\max}$ as a height bound. With the still cruder assumption that solutions form a Poisson process in logarithmic height, the probability of finding zero solutions by this scale is about 8%. Thus the failure is mildly atypical—enough to make 114 an interesting target—but not in serious tension with the positive-density conjecture.

Heuristics: why new ideas are likely needed

A stronger probabilistic model makes the computational barrier explicit. Suppose that, after conditioning on no solution up to height B_0, counts in subsequent logarithmic intervals are approximately independent Poisson variables with intensity ρ_{114}. If H denotes the first-solution height, then

$$\Pr(H > TB_0 \mid H > B_0) \approx T^{-\rho_{114}}. \tag{7}$$

Two consequences are

$$T_{\text{median}} = 2^{1/\rho_{114}} \approx 1.41 \times 10^5, \qquad T_{\text{one event}} = e^{1/\rho_{114}} \approx 2.69 \times 10^7. \tag{8}$$

The second number is the multiplicative increase in height needed to add just one to the *expected* solution count. If one summarizes the completed searches by the effective height in (6), the conditional median scale is about 4×10^{23}, while one additional expected event is not accumulated until roughly 8×10^{25}. These figures should not be read as precise forecasts: the Poisson assumption is substantially stronger than Heath-Brown's conjecture. Their robust message is that the conditional distribution is extremely broad and multiplicative, not concentrated near a single anticipated size.

This is precisely where the known algorithm meets a scaling wall. Its running time is essentially linear in the main search-height parameter up to slowly growing factors, whereas the expected number of solutions grows only as $\log B$. Under a comparably optimized continuation, gaining one additional expected event requires increasing the effective height by about 2.7×10^7; before constant-factor improvements, the near-linear complexity points to a comparable multiplicative increase in work.

Existing arithmetic filters are also unusually weak for this value. In the cubic-reciprocity sieve of [3], about 96.2% of the residue classes for 114 that survive the basic local conditions also survive the extra reciprocity test; the corresponding filter removes only about 3.8% at that stage. Constant-factor engineering remains useful, but it is unlikely by itself to bridge the heuristic gap. A successful attack may need a stronger necessary condition, a way to rank the promising values of d far more sharply, a number-field mechanism that isolates the correct norm, or a search whose cost is genuinely sublinear.

Why this is a suitable open problem. It is locally soluble and has positive conjectural density; analogous heuristics are strongly supported in aggregate; a proposed answer is instantly and independently checkable; there is a mature public baseline implementation; and the broader sums-of-three-cubes problem has attracted substantial popular interest with 114 the most-wanted value for the last 5 years [10]. At the same time, the failed searches, the small value of ρ_{114}, the weak reciprocity filtering, and the near-linear cost of exhaustive extension all indicate that success is likely to require more than simply allocating modestly more compute.

References

[1] P. Bacik, J. Nieuwveld, J. Ouaknine, M. Vahanwala, M. Venkatesh, and E. R. Wieser, "On variable-bounded non-linear expansions of Presburger arithmetic," arXiv:2605.16985 (2026).

[2] A. R. Booker, "Cracking the problem with 33," *Research in Number Theory* **5** (2019), Art. 26.

[3] A. R. Booker and A. V. Sutherland, "On a question of Mordell," *Proceedings of the National Academy of Sciences* **118** (2021), e2022377118, doi:10.1073/pnas.2022377118.

[4] J. Grantham and P. G. Walsh, "Representing integers as a sum of three cubes," arXiv:2211.12149 (2022).

[5] B. Grechuk and A. Ratcliffe, "On the shortest open cubic equations," arXiv:2603.29831 (2026).

[6] D. R. Heath-Brown, "The density of zeros of forms for which weak approximation fails," *Mathematics of Computation* **59** (1992), 613–623.

[7] D. R. Heath-Brown, W. M. Lioen, and H. J. J. te Riele, "On solving the Diophantine equation $x^3+y^3+z^3=k$ on a vector computer," *Mathematics of Computation* **61** (1993), 235–244.

[8] B. Poonen, "Undecidability in number theory," *Notices of the American Mathematical Society* **55** (2008), 344–350.

[9] S. Siksek, "Sums of integer cubes," *Proceedings of the National Academy of Sciences* **118** (2021), e2103697118, doi:10.1073/pnas.2103697118.

[10] Wikipedia contributors, "Sums of three cubes," *Wikipedia, The Free Encyclopedia*, `https://en.wikipedia.org/wiki/Sums_of_three_cubes` (accessed July 30, 2026).

Public implementation of the Booker–Sutherland search: `https://github.com/AndrewVSutherland/SumsOfThreeCubes`.