FrontierMath Open Problems

Epoch AI

Stretched Littlewood-Richardson coefficients

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be an integer partition of n, $\mu = (\mu_1, \mu_2, \ldots)$ be an integer partition of m and $\nu = (\nu_1, \nu_2, \ldots)$ be an integer partition of $n - m$. The irreducible polynomial representations of the general linear group are the Weyl modules and are indexed by partitions. The Littlewood-Richardson (LR) coefficients $c^{\lambda}_{\mu\nu}$ are the multiplicities of irreducible GL module V_λ in the tensor product of two other irreducible GL modules $V_\mu \otimes V_\nu$, namely they are defined as

$$V_\mu \otimes V_\nu = \bigoplus_{\lambda} V_\lambda^{\oplus c^{\lambda}_{\mu\nu}}.$$

They are also structure constants in the ring of symmetric functions (when multiplying Schur functions and expanding them in that basis):

$$s_\mu(x_1, x_2, \ldots)s_\nu(x_1, x_2, \ldots) = \sum_{\lambda} c^{\lambda}_{\mu\nu} s_\lambda(x_1, x_2, \ldots).$$

They have a combinatorial interpretation as counting certain types of semi-standard Young tableaux. For all the different formulas and context see [10, 9]. Overall, they are central quantities in algebraic combinatorics and posses remarkable properties that are still not fully understood. The LR coefficients are also important in algebraic geometry (intersection of Schubert cells), physics and geometric complexity theory (GCT), see [7] for an overview of these aspects.

The breakthrough proof of the saturation conjecture [6] introduced a polytope (the hive polytope), whose number of integer points is the LR coefficient. This immediately implied that the stretched LR coefficients $c^{t\lambda}_{t\mu,t\nu}$ are a quasi-polynomial in t. In [5] King–Tollu-Tomazet conjectured that the stretched Littlewood-Richardson coefficients $c^{t\lambda}_{t\mu,t\nu}$ are polynomials in t with nonnegative coefficients, that is

$$c^{t\lambda}_{t\mu,t\nu} = a_0(\lambda, \mu, \nu)t^d + a_1(\lambda, \mu, \nu)t^{d-1} + \cdots + a_d(\lambda, \mu, \nu).$$

The polynomiality property has been since proven, see [2, 8, 11]. Very special cases of the integrality of the coefficients were done in [4] in the case when $d = 1$, which corresponds to $c^{\lambda}_{\mu\nu} = 2 \Longrightarrow c^{t\lambda}_{t\mu t\nu} = t + 1$. The nonnegativity of the coefficients still stands with very little progress in either direction. Our goal is to disprove this conjecture.

Open problem. Find integer partitions λ, μ, ν, such that $|\lambda| = |\mu| + |\nu|$ and such that the polynomial

$$c^{t\lambda}_{t\mu,t\nu} = a_0(\lambda, \mu, \nu)t^m + \cdots + a_{d-1}(\lambda, \mu, \nu)t + a_d(\lambda, \mu, \nu)$$

has a coefficient $a_i(\lambda, \mu, \nu) < 0$.

Special cases of this problem cover the Ehrhart polynomial of the Chen-Robbins-Yuen polytope (when $\mu = \nu = (n-1, n-2, \ldots, 1)$ and $\lambda = 2n-1, 2(n-1), 2(n-2), \ldots, 2, 1)$, as well as the stretched Kostka numbers (alternatively Gelfand-Tsetlin patterns), neither of which is known to have nonnegative coefficients. The degree of the polynomial is not generally known, but we have $d \leq \binom{\ell(\lambda)+1}{2}$, where $\ell(\lambda)$ is the number of nonzero parts of the partition λ.

We are looking for the 3 partitions λ, μ, ν which solve this open problem, i.e. for which there is some i and $a_i(\lambda, \mu, \nu) < 0$. The attached code verifies the answer, i.e. given the three partitions it computes the polynomial and checks if indeed some coefficient is < 0.

One way to compute the polynomial $c^{t\lambda}_{t\mu t\nu}$ is as follows. First, assume that $c^{\lambda}_{\mu\nu} \geq 1$, otherwise the stretched LR is always 0. Then

$$c^{t\lambda}_{t\mu,t\nu} = \#\{H(t\lambda, t\mu, t\nu) \cap \mathbb{Z}^{\binom{k+1}{2}}\} = i(H(\lambda, \mu, \nu), t),$$

where $i(P,t) = \#\{tP \cap \mathbb{Z}^{dim(P)}\}$ is the Ehrhart (quasi)polynomial of P. The polytope H is the hive polytope defined as

Index set: $\Delta_n = \{(i,j,k) \in \mathbb{Z}^3_{\geq 0} : \ i+j+k=n\}, \quad h_{i,j,k} \in \mathbb{R}.$

Boundary data (fix a basepoint and prescribe edge slopes):

$$h_{0,n,0} = 0,$$
$$h_{i,n-i,0} - h_{i-1,n-i+1,0} = \lambda_i \quad (i=1,\ldots,n),$$
$$h_{0,n-r,r} - h_{0,n-r+1,r-1} = \mu_r \quad (r=1,\ldots,n),$$
$$h_{r,0,n-r} - h_{r-1,0,n-r+1} = \nu_r \quad (r=1,\ldots,n).$$

Rhombus (discrete concavity) inequalities:

$$h_{i,j,k} + h_{i+1,\,j,\,k-1} \ \geq \ h_{i+1,\,j-1,\,k} + h_{i,\,j+1,\,k-1} \quad \text{for } j,k \geq 1,\ i \geq 0,\ i+j+k=n,$$
$$h_{i,j,k} + h_{i-1,\,j+1,\,k} \ \geq \ h_{i,\,j+1,\,k-1} + h_{i-1,\,j,\,k+1} \quad \text{for } i,k \geq 1,\ j \geq 0,\ i+j+k=n,$$
$$h_{i,j,k} + h_{i,\,j-1,\,k+1} \ \geq \ h_{i-1,\,j,\,k+1} + h_{i+1,\,j-1,\,k} \quad \text{for } i,j \geq 1,\ k \geq 0,\ i+j+k=n.$$

Why the conjecture might be false. Positivity in combinatorics is a very rare and special phenomenon. Usually such polynomials exhibit positivity as part of some deeper structure, either algebraic or geometric. Many Ehrhart or order polynomials for very simple polytopes are not positive. For example the order polynomial of the poset consisting of an antichain and one maximal element does not have positive coefficients. Other counterexamples can be found in [1, 3]. Overall, there is no intrinsic reason for the positivity to hold. The main reason why no counterexample has been found so far is that the LR coefficients are still not very efficient to compute, the parameter space is large depending on three partitions, and it is hard going past $\ell(\lambda) \geq 6$ or so. It would be reasonable to do a search on partitions with $\ell(\lambda) \leq 20$.

References

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