

$$1. z^2 + y^2z - z + x^3 + 2 = 0$$

A clean infinite family is:

$$y = s + 2, \quad z = -31s^2 + 4s - 1, \quad x = us$$

Then

$$z^2 + y^2z - z + x^3 + 2 = s^3, (u^3 + 930s - 368).$$

So any integers u, s with

$$u^3 + 930s = 368$$

give a solution. Equivalently, if $u^3 \equiv 368 \pmod{930}$, then

$$s = \frac{368 - u^3}{930} \in \mathbb{Z}$$

and

$$(x, y, z) = (us, s + 2, -31s^2 + 4s - 1)$$

is an integer solution.

Three convenient congruence classes are

$$u \equiv 542, 602, 902 \pmod{930},$$

so taking $u_1 = 930 \cdot 10^{11} + 542$, $u_2 = 930 \cdot 10^{11} + 602$, $u_3 = 930 \cdot 10^{11} + 902$ gives three explicit solutions:

$$(x_1, y_1, z_1) = (-80435700001875103200016392031200063687998400092792568, -8649000000151218000000881$$

$$(x_2, y_2, z_2) = (-80435700002082679200020222143200087266846400141221976, -8649000000167958000001087$$

$$(x_3, y_3, z_3) = (-80435700003120559200045399103200293548286400711775416, -8649000000251658000002440$$

Each satisfies

$$z^2 + y^2z - z + x^3 + 2 = 0,$$

the three x -values are distinct, and each has $|x| > 10^{50}$.

1.1. Here's the construction. Start from

$$z^2 + y^2z - z + x^3 + 2 = 0$$

and rewrite it as

$$x^3 = -(z^2 + (y^2 - 1)z + 2).$$

So the goal is to make the right-hand side an obvious cube.

I looked for a parameter s so that the right-hand side becomes $s^3 \times$ (something simple).

A natural ansatz is

$$y = s + a, \quad z = bs^2 + cs + d, \quad x = us$$

with integers a, b, c, d, u .

If we set

$$F(s) = z^2 + (y^2 - 1)z + 2,$$

then expanding gives

$$\begin{aligned} F(s) = & (b^2 + b)s^4 + (2ab + 2bc + c)s^3 \\ & + (a^2b + 2ac + 2bd - b + c^2 + d)s^2 \\ & + (a^2c + 2ad + 2cd - c)s \\ & + (a^2d + d^2 - d + 2). \end{aligned}$$

To force $F(s)$ to be divisible by s^3 , I made the constant, linear, and quadratic coefficients vanish. That gives:

$$\begin{cases} a^2d + d^2 - d + 2 = 0, \\ a^2c + 2ad + 2cd - c = 0, \\ a^2b + 2ac + 2bd - b + c^2 + d = 0. \end{cases}$$

Now choose a so the first equation has integer solutions in d . That equation is

$$d^2 + (a^2 - 1)d + 2 = 0,$$

whose discriminant is

$$(a^2 - 1)^2 - 8 = a^4 - 2a^2 - 7.$$

Trying small a , the nice choice is $a = 2$, because then the discriminant is

$$2^4 - 2 \cdot 2^2 - 7 = 16 - 8 - 7 = 1.$$

So d is integral, and in fact

$$d^2 + 3d + 2 = 0 \quad \Rightarrow \quad d = -1 \text{ or } -2.$$

I took $d = -1$. Then the next two equations become:

$$4c - 4 + c(2(-1)) - c = c - 4 = 0 \quad \Rightarrow \quad c = 4,$$

and then

$$4b + 2(2)(4) + 2b(-1) - b + 4^2 - 1 = 0$$

which simplifies to

$$b + 31 = 0 \quad \Rightarrow \quad b = -31.$$

So this produces

$$y = s + 2, \quad z = -31s^2 + 4s - 1.$$

Now substitute back:

$$z^2 + (y^2 - 1)z + 2 = 930s^4 - 368s^3 = s^3(930s - 368).$$

Therefore

$$x^3 = -s^3(930s - 368) = s^3(368 - 930s).$$

At that point the cube structure is visible: if I set

$$x = us,$$

then I only need

$$u^3 = 368 - 930s,$$

or equivalently

$$u^3 + 930s = 368.$$

So the whole family is

$$s = \frac{368 - u^3}{930}, \quad y = s + 2, \quad z = -31s^2 + 4s - 1, \quad x = us,$$

provided $u^3 \equiv 368 \pmod{930}$.

That congruence has three residue classes mod (930):

$$u \equiv 542, 602, 902 \pmod{930}.$$

So for any integer k ,

$$u = 930k + 542, \quad 930k + 602, \quad 930k + 902$$

gives an integer s , hence an integer solution (x, y, z) . That is why there are infinitely many such solutions, and why it is easy to make $|x|$ huge: for large u ,

$$x = u \cdot \frac{368 - u^3}{930} \sim -\frac{u^4}{930},$$

so $|x|$ grows like u^4 .

There is also a second nice branch from the other root $d = -2$, which gives

$$y = s + 2, \quad z = 30s^2 - 8s - 2,$$

and leads to another infinite family.

$$2. z^2 + y^2z + x^3 + x + 1 = 0$$

One infinite family of integer solutions is

$$x(n) = -108n^4 - 144n^3 - 96n^2 - 32n - 6,$$

$$y(n) = 36n^3 + 36n^2 + 14n + 2,$$

$$z(n) = 648n^6 + 1296n^5 + 1368n^4 + 864n^3 + 362n^2 + 92n + 13.$$

A direct substitution shows

$$z(n)^2 + y(n)^2z(n) + x(n)^3 + x(n) + 1 = 0$$

identically for every integer n .

So taking $n = 10^{13}, 10^{13} + 1, 10^{13} + 2$ gives three distinct integer solutions:

* For $n = 10^{13}$,

$$(-1080000000000144000000000096000000000032000000000006, ; 36000000000003600000000000140000000000$$

* For $n = 10^{13} + 1$,

$$(-10800000000000576000000000117600000000010880000000000386, ; 36000000000014400000000001940000000000$$

* For $n = 10^{13} + 2$,

$$(-108000000000100800000000035520000000005600000000003334, ; 3600000000002520000000000590000000000$$

The x -values are all different, and each has 55 digits, so in particular $|x| > 10^{50}$.

2.1. Here's the construction. The clean way to see it is to start from a simpler equivalent family and then note that the one I wrote before is just a shifted version of it.

A cleaner parametrization is

$$x(t) = -108t^4 - 24t^2 - 2, \quad y(t) = 36t^3 + 2t, \quad z(t) = 648t^6 + 288t^4 + 50t^2 + 3.$$

For every integer t , these are integers and satisfy

$$z^2 + y^2z + x^3 + x + 1 = 0.$$

The family I gave earlier is exactly this one after the change of variable

$$t = n + \frac{1}{3}.$$

If you expand $x(t), y(t), z(t)$ with $t = n + \frac{1}{3}$, you get the formulas I previously wrote.

How I found it

I rewrote the equation as

$$z(z + y^2) = -(x^3 + x + 1).$$

So the goal became:

- make $x^3 + x + 1$ factor nicely, and
- arrange that the two factors differ by a square, because $z + y^2$ is just z plus a square.

Step 1: choose a degree pattern

To balance the sizes of the terms, I looked for a one-parameter polynomial family of the form

$$x = at^4 + bt^2 + c, \quad y = pt^3 + qt, \quad z = rt^6 + st^4 + ut^2 + v.$$

This is natural because then

- x^3 has degree 12,
- z^2 has degree 12,
- y^2z also has degree 12.

So the degrees match.

Step 2: expand and match coefficients

Substitute those into

$$z^2 + y^2z + x^3 + x + 1 = 0$$

and equate coefficients of $t^0, t^2, t^4, \dots, t^{12}$.

One nice solution of that coefficient system is

$$(a, b, c, p, q, r, s, u, v) = (-108, -24, -2, 36, 2, 648, 288, 50, 3).$$

That gives exactly

$$x = -108t^4 - 24t^2 - 2, \quad y = 36t^3 + 2t, \quad z = 648t^6 + 288t^4 + 50t^2 + 3.$$

Step 3: the hidden factorization

Once those coefficients appeared, the reason it works became transparent.

Define

$$A = 648t^6 + 144t^4 + 18t^2 + 1, \quad B = 648t^6 + 288t^4 + 50t^2 + 3.$$

Then the key identities are

$$x^3 + x + 1 = -3AB$$

and

$$y^2 = 3A - B.$$

Indeed,

$$(36t^3 + 2t)^2 = 1296t^6 + 144t^4 + 4t^2 = 3A - B.$$

Now set

$$z = B.$$

Then

$$z + y^2 = B + (3A - B) = 3A,$$

so

$$z(z + y^2) = B \cdot 3A = 3AB.$$

Therefore

$$z^2 + y^2z = 3AB = -(x^3 + x + 1),$$

which is exactly

$$z^2 + y^2z + x^3 + x + 1 = 0.$$

So the family works for every integer (t).

Why the earlier family looked messier

The family I wrote before was

$$x(n) = -108n^4 - 144n^3 - 96n^2 - 32n - 6, \quad y(n) = 36n^3 + 36n^2 + 14n + 2, \quad z(n) = 648n^6 + 1296n^5 + 1368n^4 +$$

That is just the same clean family with

$$t = n + \frac{1}{3}.$$

For example,

$$-108\left(n + \frac{1}{3}\right)^4 - 24\left(n + \frac{1}{3}\right)^2 - 2 = -108n^4 - 144n^3 - 96n^2 - 32n - 6.$$

So conceptually, I found the clean t -family first, and the earlier one is just a shifted re-expression of it.