FrontierMath Open Problems

Epoch AI

Ramsey numbers for book graphs

Problem statement

Given graphs G and H, the *Ramsey number* $R(G, H)$ is the smallest n such that every red/blue coloring of the edges of K_n contains a red copy of G or a blue copy of H. In general, computing this value precisely is extremely difficult. The *book graph* B_n is the graph $K_2 + \overline{K_n}$, where + denotes the graph join. That is, B_n consists of n triangles that share a common edge. The first systematic study of book Ramsey numbers was done in 1978 by Rousseau and Sheehan [5]. They proved, among other things, that $R(B_{n-1}, B_n) \leq 4n - 1$ for all n, but did not produce constructions to give a lower bound. Recent work by Wesley [6] and Lidický, McKinley, Pfender, and Van Overberghe [3] shows that this bound is sharp when $2n - 1$ is a prime power congruent to 1 modulo 4 and for all $n \leq 21$. The lower bound colorings are "2-block circulant" graphs and particular generalizations of Paley graphs. Given this computational evidence, we are optimistic that Rousseau and Sheehan's bound is sharp for all n.

Problem 1. *Show that* $R(B_{n-1}, B_n) = 4n - 1$ *for all* n.

Significance to field

There is a huge body of work in computing small Ramsey numbers that spans over a hundred pages in the dynamic survey of Radziszowski [4]. The "classical" numbers $R(K_s, K_t)$ are the most notable, but Ramsey numbers involving cycles, books, and wheels have also attracted significant attention. In particular, Ramsey numbers for book graphs are connected to strongly regular graphs, and recent work has studied the asymptotics of $R(B_n, B_n)$ [1, 2].

Finding a general construction for $R(B_{n-1}, B_n)$ would be of interest to computational graph theorists. Such a construction would likely be useful for proving bounds for other values of $R(B_m, B_n)$, and possibly for other general Ramsey numbers as well. Even having more data for larger n would be helpful in understanding why Rousseau and Sheehan's bound is tight and the graphs produced would be nice contributions to databases such as House of Graphs.

Relative difficulty

There has been a moderate amount of effort put into computing values of $R(B_{n-1}, B_n)$, but it would be reasonable for a computational graph theory expert to extend the computational methods of [6] or [3] to, say, $n \leq 30$. This would likely require days to weeks of computation time. There may be a nice construction for all n, but it has eluded several researchers in this area, so it is probably not obvious. It is also not clear whether the best approach is to look for similar block circulant constructions, or to search for something entirely different.

References

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[3] Bernard Lidický et al. *Small Ramsey numbers for books, wheels, and generalizations*. arXiv:2407.07285. 2024. arXiv: 2407.07285. URL: https://arxiv.org/abs/2407.07285.

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