FrontierMath Open Problems

Epoch AI

The absolute Galois group of $\mathbb{Q}_2$

Problem statement

Give a presentation for the absolute Galois group of the field of 2-adic numbers as a profinite group.

Overview

The absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ of a field K is the projective limit of all of the finite Galois groups $\mathrm{Gal}(E/K)$. It packages together the information about all finite extensions, and studying the Galois group of the rational field $\mathbb{Q}$ is a central problem in algebraic number theory. One method for approaching this group is to study the analogous Galois groups of p-adic fields $\mathbb{Q}_p$. In this case, there is an explicit presentation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for $p > 2$, described in Section . For $p = 2$, we have a description for the absolute Galois group of some extensions of $\mathbb{Q}_2$, but not for $\mathbb{Q}_2$ itself. Finding such a description would fill in a gap in our explicit understanding of Galois groups, and would have consequences in terms of counting p-adic fields with a given Galois group.

Background

We describe the presentation for known $p > 2$. Let h be a $(p-1)$st root of unity in $\mathbb{Z}_p$. Let $\pi = \pi_p$ be the element of $\hat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$ with coordinate 1 in the $\mathbb{Z}_p$-component and 0 in the $\mathbb{Z}_\ell$ components for $\ell \neq p$. Then for x, y in a profinite group[2], set

$$\langle x, y\rangle = (x^{h^{p-1}} y x^{h^{p-2}} y \cdots x^h y)^{\frac{\pi}{p-1}}.$$

Write $[x, y]$ for the commutator $xyx^{-1}y^{-1}$.

Theorem 2 ([1]*Thm. 7.5.14)**.** *Let $p \neq 2$. The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p))$ is isomorphic to the profinite group with 4 generators σ, τ, x_0, x_1, subject to the following conditions and relations.*

1. *The closed subgroup topologically generated by x_0 and x_1 is normal and pro-p.*
2. *The elements σ, τ satisfy the tame relation*
$$\tau^\sigma = \tau^q.$$
3. *The generators satisfy the wild relation*
$$x_0^\sigma = \langle x_0, \tau\rangle x_1^p [x_1, y_1],$$
where y_1 is defined below.

In order to define y_1, let $\mathbb{Q}_p^t$ be the maximal tamely ramified extension of $\mathbb{Q}_p$ and define $\beta : \mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$ by setting $\beta(\sigma) = 1$ and $\beta(\tau) = h$. For ρ in the subgroup generated by σ and τ and an arbitrary x, set

$$\{x, \rho\} = (x\rho^2 x^{\beta(\rho)} \rho^2 \ldots x^{\beta(\rho^{p-2})} \rho^2)^{\frac{\pi}{p-1}}.$$

Similar to π_p above, let $\pi_2 \in \hat{\mathbb{Z}}$ have a 1 in the $\mathbb{Z}_2$-component and 0 in every other component, and set $\tau_2 = \tau^{\pi_2}$ and $\sigma_2 = \sigma^{\pi_2}$. Set

$$y_1 = x_1^{\tau_2^{p+1}} \{x_1, \tau_2^{p+1}\}^{\sigma_2 \tau_2^{(p-1)/2}} \{\{x_1, \tau_2^{p+1}\}, \sigma_2 \tau_2^{(p-1)/2}\}^{\sigma_2 \tau_2^{(p+1)/2} + \tau_2^{(p+1)/2}}.$$

[2] See [2], especially sections 3.3 and 4.1, for relevant background on profinite groups.

Importance

Galois representations are at the core of many problems in modern number theory. There are many perspectives on Galois representations, but if you have access to a presentation of an absolute Galois group then they can be viewed very concretely. The case of $p = 2$ often needs to be handled separately in number theory, and global applications often require the ability to handle all primes.

Difficulty

The last major work on finding presentations for p-adic fields was done in the 1980s. To a large extent, I believe that the difficulty is that the answer is likely to be messy, as evidenced by the complicated relation needed in the case that p is odd. Hopefully, this feature makes the problem a good target for solution by an AI since I don't expect major conceptional developments to be required.

Submitting a presentation

A submission for this problem consists of a file with the following form:

```
variables

definitions

relations
```

- Sections should be separated by single blank lines, and the `definitions` section is optional.
- The `variables` should consist of a comma separated list of variable names on one line corresponding to the generators of the presentation. The first and second generators (say σ and τ respectively) are distinguished, and must satisfy the implicit relation $\tau^\sigma = \tau^p$ (which should not be included in the `relations` section). All later generators should be contained within the p-core of the group (the intersection of all p-Sylow subgroups).
- In the `definitions` section, you may define additional variables in terms of the generators and variables defined earlier within the `definitions` section. Each line should take the form `var = expr`, where `expr` is an expression built using group operations.
- In the `relations` section, you should give the relations as expressions in terms of variables defined in the previous two sections. If there are multiple relations, give one on each line.

Expressions in profinite groups may involve exponents lying in $\hat{\mathbb{Z}}$. To specify such a constant, you should provide its reduction modulo $85667662080 = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, which is the least common multiple of the exponents of all the groups being used to verify the presentation.

References

[1] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number Fields*. 2nd ed. Berlin: Springer-Verlag, 2015.

[2] Luis Ribes and Pavel Zalesskii. *Profinite Groups*. 2nd ed. Berlin: Springer-Verlag, 2010.