# FrontierMath Open Problems

Epoch AI

## Finding Small Block Designs

Block designs are combinatorial objects with real use in experimental design, error-correcting codes, and tournament brackets. Questions about the existence of block designs are among "the oldest problems in combinatorics, dating back to work of Plücker (1835), Kirkman (1846) and Steiner (1853)" [5].

Let $[n]$ be the elements $\{1, \ldots, n\}$. Call $T$ a $q$-subset of $[n]$ if $T \subseteq [n]$ and $|T| = q$.

**Definition.** An $(n, q, r, z)$-*block design* is a set $\mathcal{S}$ of $q$-subsets of $[n]$ such that every $r$-subset of $[n]$ is contained in exactly $z$ elements of $\mathcal{S}$.

It is helpful to visualize these objects when $z = 1$. These are called *Steiner systems*. An example Steiner system is the Fano plane $(7, 3, 2)$, where each 3-subset is a line in the below picture.

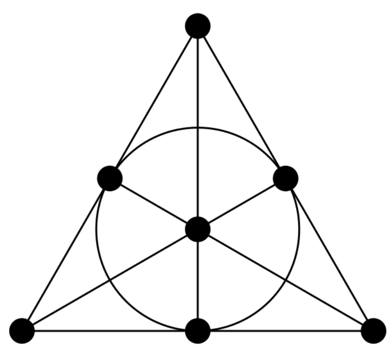

Figure 1: Fano Plane. Public domain, via Wikimedia Commons

Block designs must satisfy some divisibility rules on their parameters. (One such rule is that the number of $r$-subsets of $[q]$ must divide the number of $r$-subsets of $[n]$.) It was long conjectured, and recently proven, that these divisibility rules are actually sufficient to find designs.

**Theorem [5, 1, 2, 6] (informal).** Fix any $q, r, z$. Then an $(n, q, r, z)$-block design exists for all $n$ satisfying the divisibility rules, apart from a finite number of exceptions.

From [6], the number of exceptions is not more than $q^{O\left(k^2 q^{(r+1)}\right)}$.

This result was nicely covered by Quanta Magazine [8] and Gil Kalai [4]. It also accelerated a number of results in combinatorics and theoretical computer science; see Section 1.3 of [5]. However, it cannot answer the following question:

**Open Problem.** Fix any $q, r, z$. Find a small $n$ such that an $(n, q, r, z)$-block design exists, and return the block design.

Practical applications of block designs require explicit constructions. We have compiled lists of known block designs [3, 7]. However, for most parameters, we have no explicit block designs:

**Open Subproblem 1.** Find an $(n, q, r)$-Steiner system for any $r > 5$. (There is always a trivial design when $n = q$ or $q = r$, so assume $n > q > r$.)

It is easy to verify that a set $\mathcal{S}$ of $q$-subsets of $[n]$ is an $(n, q, r, z)$-block design when all parameters are small. We generate every $r$-subset, and show that it is contained in exactly $z$ $q$-subsets in $\mathcal{S}$. This naively takes time $n^{r+q}$, and there are faster algorithms.

Beyond Open Subproblem 1, explicit $(n, q, r, z)$-block designs could have practical applications for many "reasonable" sizes of parameters, say $n < 1000$ and $q < 50$. As a fun example, [8] explains how to win the Massachusetts Cash WinFall lottery in the fewest number of tickets using a $(46, 6, 5)$-Steiner system. It is likely that new block designs will imply new error correcting codes. I think solving Open Subproblem 1 would merit publication in a good combinatorics journal, especially with an understanding of any group structure in the construction.

# References


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[5] P. Keevash. *The Existence of Designs*. arXiv preprint. 2014. URL: https://arxiv.org/pdf/1401.3665.

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[7] K. Marwaha. *Steiner Systems List*. Online resource. URL: https://marwahaha.github.io/steinersystems/.

[8] Quanta Magazine. *150-Year-Old Math Design Problem Solved*. Online article. 2015. URL: https://www.quantamagazine.org/150-year-old-math-design-problem-solved-20150609/.