FrontierMath Open Problems

Epoch AI

KLT del Pezzo Surface in Characteristic 3 with more than 7 Singular Points

Introduction

A del Pezzo surface is an algebraic surface whose anticanonical divisor $-K_X$ is ample; equivalently, it is a two–dimensional Fano variety. Del Pezzo surfaces, or more in general Fano varieties, are central in birational geometry: by the Minimal Model Program (MMP), every algebraic variety is expected to be birational to a variety which is built up with (i) varieties of general type, (ii) Calabi–Yau (i.e. numerically trivial canonical class), or (iii) Fano varieties (i.e. negative canonical class). Thus Fano varieties form one of the three foundational building blocks in the birational classification of algebraic varieties.

Over an algebraically closed field K of characteristic 0 (e.g. $K = \mathbb{C}$), del Pezzo surfaces with mild (klt) singularities are highly constrained. Over an algebraically closed field K of positive characteristic p (e.g. $K = \mathbb{F}_p$), and especially at small primes $p = 2, 3$, a number of surprising pathologies appear (e.g. quasi–elliptic fibrations exist only for $p = 2, 3$ [2]; Kodaira and Kawamata–Viehweg vanishing may fail [5]; wild quotient and inseparable phenomena can appear [1]). These effects make the small–characteristic geometry both delicate and mathematically rich.

We propose the following:

Problem 0.1 Construct an explicit normal projective surface X over an algebraically closed field of characteristic 3 such that

(1) X is a klt del Pezzo surface;

(2) $\rho(X) = 1$ (Picard number 1);

(3) X has more than seven singular points.

Note that condition (2) is technical but it is very standard in birational geometry as Fano varieties of Picard number 1 are exactly one of the three building blocks in birational geometry. The answer should be a concrete presentation of X (e.g. as a weighted projective hypersurface, or a complete intersection, or a global quotient Y/G of a smooth del Pezzo Y).

State of the art

Small characteristic exhibits phenomena absent in characteristic 0 (e.g. over $\mathbb{C}$). In characteristic 2, Keel–McKernan [3, §9] constructed klt del Pezzo surfaces of Picard number one and with arbitrarily many singular points. In characteristic 3, all currently known constructions achieve at most 7 singular points on klt del Pezzo surfaces of Picard number one [4]; as far as I know, no example with more than 7 singular points is known. By contrast, in characteristic zero [3] and for $p > 3$ [4], one can have at most four singular points, showing how exceptional $p = 2, 3$ are. In addition, the possible singularities that can appear in a klt del Pezzo surface in characteristic zero have been classified.

A construction in characteristic 3 with more than 7 singularities would uncover a new small–characteristic phenomenon; a failure to find such an example (together with new obstructions) would be, in my opinion, very surprising as it would make the case $p = 2$ as uniquely wild. Either outcome would matter for the broader program of understanding Fano varieties and the MMP in positive characteristic (e.g. see [6]).

Solution Verifier

Rather than prescribing a single presentation, we present three distinct construction methods for the target surface; in all cases the resulting models can be verified in Macaulay2.

Method A - Global quotients.

Start from a smooth del Pezzo surface Y (more specifically, a smooth complete intersection of bidegree $(2, 2)$ in $\mathbb{P}^4$) and a finite tame cyclic group G acting freely in codimension 1. Then $X = Y/G$ is a klt del Pezzo; the singularities correspond to isolated fixed points on Y.

Method B - Weighted models.

Realize X as a weighted hypersurface/complete intersection in a weighted projective space $\mathbb{P}(w)$. Singularities arise from intersections with ambient singular strata. Quasi-smoothness ensures we only have tame cyclic quotient points.

Method C - blow-ups.

Provide a Hirzebruch–Jung (HJ) list of exceptional chains to extract and then contract, starting from $\mathbb{P}^1 \times \mathbb{P}^1$ (this is the method used to construct a klt del Pezzo surface in characteristic three with 7 singular points). The tool symbolically computes discrepancies, checks klt, and reports $\rho(X)$ and K_X^2.

Variants

- Reproduce the Keel–McKernan family in characteristic two with a prescribed number of singular points (currently constructed with Method C); verify via Method A or B.
- Construct a one–parameter family in characteristic 3, where $\#\mathrm{Sing}(X)$ grows with the parameter while maintaining $\rho = 1$ and klt.
- Find new configurations of singularities that can appear in a klt del Pezzo surface of characteristic $p = 2$ and $p = 3$.

References

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