FrontierMath Open Problems

Epoch AI

Deformations of Curvilinear Algebras to Monomial Algebras

The Deformation Problem

Let k be a field of characteristic zero. We are interested in explicit deformations between finitely generated local k-algebras, specifically from curvilinear (also called Morin or A_n) types to monomial algebras. Consider the monomial algebra

$$A = k[x, y, z]/(x^8, y^8, z^8, xy, xz, yz).$$

It has 4 generators and forms a 22-dimensional vector space over k corresponding to a "spider"-shaped box-arrangement with 3 legs of length 7. The goal is to deform the curvilinear algebra $k[t]/(t^{22})$ into A.

Problem: Construct an explicit flat deformation of $k[t]/(t^{22})$ to A. That is, build a family of algebras $k[x_1, \ldots, x_s, \epsilon]/I$ over $k[\epsilon]$ whose fiber over $\epsilon = 0$ is A, and whose generic fiber ($\epsilon \neq 0$) is isomorphic to $k[t]/(t^{22})$.

Illustration: A simpler version of this problem is deforming

$$k[t]/(t^3) \quad \text{to} \quad k[x, y]/(x, y)^2.$$

Here, the deformation can be made explicit: one considers the Rees algebra associated to the t-adic filtration on $k[t]/(t^3)$. Introducing variables x and y corresponding to the classes of t and t^2 respectively, we can define a family

$$B = k[x, y, \epsilon]/(x^2, xy, y^2 - \epsilon x).$$

When $\epsilon \neq 0$, solving $y^2 = \epsilon x$ shows that x and y are dependent, and the algebra is generated by y alone, leading to an isomorphism with $k[t]/(t^3)$. When $\epsilon = 0$, the relations become $x^2 = xy = y^2 = 0$, yielding the algebra $k[x, y]/(x, y)^2$.

This shows the subtlety: while $k[t]/(t^3)$ is generated by powers of a single t, its limit breaks into two variables x, y with quadratic relations. Keeping control of such deformations is computationally delicate, especially in higher embedding dimension and multiplicity.

Background and Context

The study of deformations of Artinian algebras is central to the geometry of the Hilbert scheme of points, $\mathrm{Hilb}^k(\mathbb{C}^n)$. The curvilinear component $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ plays a distinguished role: it parametrizes subschemes that are locally isomorphic to $\mathrm{Spec}(k[t]/(t^k))$ inside $\mathbb{C}^n$.

It is a classical result, first proved in Briançon's PhD thesis [4], that for monomial ideals in two variables (i.e., plane monomial ideals), every such ideal is the deformation of the curvilinear ideal of the same dimension. In other words, every monomial subscheme of $\mathbb{C}^2$ lies in the curvilinear component of $\mathrm{Hilb}^k(\mathbb{C}^2)$. More recently, the general n-variable version was established:

Theorem 1 ([2])**.** *Every monomial ideal in* $\mathrm{Hilb}^k(\mathbb{C}^n)$ *lies in the curvilinear component* $\mathrm{CHilb}_0^k(\mathbb{C}^n)$*. Consequently, any monomial subscheme of* $\mathbb{C}^n$ *can be deformed into a curvilinear subscheme.*

Our proof in [2] relies on the test curve model of the curvilinear Hilbert scheme. For our method to work, we need to increase the embedding dimension. However, constructing explicit deformations remains a highly nontrivial task, especially for "spider"-shaped monomial ideals such as in the proposed problem.

Explicit constructions are crucial for applications: they provide local coordinates for the Hilbert scheme around singular points, enable effective computations in deformation theory, and allow testing broader conjectures about singularities and moduli. They also play a central role in enumerative geometry, particularly in intersection theory of the Hilbert scheme via equivariant localization [3, 1].

References

[1] G. Bérczi. *Tautological Integrals on Hilbert Scheme of Points I.* arXiv:2303.14807. 2023.

[2] G. Bérczi and J. Svendsen. *Fixed Point Distribution on Hilbert Scheme of Points.* arXiv:2306.11521. 2023.

[3] G. Bérczi and A. Szenes. *Multiple-Point Residue Formulas for Holomorphic Maps.* arXiv:2112.15502. 2021.

[4] J. Briançon. "Description de $\mathrm{Hilb}^n(\mathbb{C}^2)$". PhD thesis. Université de Nice, 1972.