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Epoch AI

Degree vs. Sensitivity for Boolean Functions: Toward Explicit Separations Beyond the Kushilevitz Barrier

Problem statement

Construct a multilinear polynomial $P : \mathbb{R}^n \to \mathbb{R}$ of degree $\deg(P) > 1$ (for some $n > 1$) such that

(i) P is Boolean on the cube, i.e. $P(x) \in \{0, 1\}$ for all $x \in \{0, 1\}^n$;

(ii) We have

$$\begin{aligned} &|P(0, \dots, 0) - P(1, 0, \dots, 0)| + |P(0, \dots, 0) - P(0, 1, 0 \dots, 0)| + \dots \\ &+ |P(0, \dots, 0) - P(0, 0, \dots, 0, 1)| \ = \big(\deg(P)\big)^{\alpha} \end{aligned} \tag{1}$$

for some exponent $\alpha > \frac{\log 6}{\log 3} \approx 1.63$.

Equivalently, the goal is to construct a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ with

$$s(f) = \big(\deg(f)\big)^{\alpha} \qquad \text{for some } \alpha > \tfrac{\log 6}{\log 3}.$$

where $s(f)$ represents max-sensitivity of the boolean function f, i.e., $\max_{x \in \{0,1\}^n} \sum_{j=1}^{n} |f(x) - f(x^j)|$, where x^j is the same vector as x, except j'th bit is flipped. (As is standard, by negating coordinates one may assume the maximum sensitivity is attained at $x = (0, \dots, 0)$, and this does not change $\deg(f)$.)

Background and significance

A central theme in the analysis of Boolean functions is the relationship among the many complexity measures associated with f, such as degree, (block) sensitivity, decision-tree complexities, certificate complexity, and others. Over the years these measures were shown to be polynomially related, culminating in Huang's resolution of the *Sensitivity Conjecture*.[1] See [6, 1, 7].

Known lower and upper bounds between $s(f)$ and $\deg(f)$. It is now known that

$$s(f) \ \geq \ \sqrt{\deg(f)} \qquad \text{for all Boolean } f\,, \tag{2}$$

and this bound is tight. On the other side we have

$$s(f) \ \leq \ \big(\deg(f)\big)^2 \qquad \text{for all Boolean } f\,, \tag{3}$$

see (e.g.) [6, 7, 5, 12]. It is not known whether quadratic upper bound is tight (up to constants). A 2021 preprint of Proskurin [8] shows that

$$s(f) \leq \frac{1}{\sqrt{10} - 2} (\deg(f))^2$$

where $\frac{1}{\sqrt{10}-2} \approx 0.86\ldots$.

[1] Huang proved $s(f) \geq \sqrt{\deg(f)}$ settling a question of Nisan and Szegedy.

Best explicit separation. The best *explicit* separation currently known between degree and sensitivity is due to Kushilevitz and is achieved by the following function

$$h(z_1,\dots,z_6) = \sum_{i=1}^{6} z_i - \sum_{1\le i<j\le 6} z_i z_j + \big(z_1z_3z_4 + z_1z_2z_5 + z_1z_4z_5 + z_2z_3z_4 + z_2z_3z_5 + z_1z_2z_6 + z_1z_3z_6 + z_2z_4z_6 + z_3z_5z_6 + z_4z_5z_6\big), \tag{4}$$

In this example $s(h) = 6$ and $\deg(h) = 3$, yielding

$$s(h) = \big(\deg(h)\big)^{\log_3 6} = \big(\deg(h)\big)^{\log(6)/\log(3)}.$$

This is the largest exponent α currently achieved by an explicit Boolean function [5, Example 5.4]; see also [12] for background.

Why this problem matters. An explicit construction with exponent $\alpha > \log(6)/\log(3)$ would strictly improve the best known separation between a *local* measure $s(f)$ and a *global* algebraic measure $\deg(f)$), and—by powering/composition—would immediately give an infinite family witnessing the same exponent. Such constructions have historically seeded advances across communication complexity, approximate degree, and learning lower bounds [4, 2, 11].

State of the art and obstacles

Two analytic tools repeatedly appear in sharp degree–sensitivity bounds: (a) symmetrization (reducing to a univariate profile on Hamming layers) and (b) inequality machinery from approximation theory (e.g. Ehlich–Zeller and Markov-type bounds) [3, 9, 10]. The preprint [8] shows how to combine symmetrization, discrete interpolation, and linear programming to *pin down a unique univariate symmetrized profile* that any hypothetical fully sensitive degree-4 function on $n = 10$ variables must have. This tantalizing "$n = 10, \deg = 4$" base, if it existed, would beat the composition exponent, as it would give $s(f) = 10$ and $\deg(f) = 4$, hence after powering $s = \Theta(n)$ and $\deg(f) = n^{\log_{10} 4} \approx n^{0.602}$, improving the exponent $\log_3 2 \approx 0.630$ implicit in the 3-variable, degree-2 base and also beating $\log_6 3 \approx 0.613$ coming from the Kushilevitz base.

Despite substantial search (including modern computational exploration), no such $n = 10, \deg = 4$ base is known. In fact, enforcing natural symmetry subclasses (cyclic/dihedral on [10], standard pair-wreath symmetries, and others) quickly leads to incompatibilities in the required local incidences across Hamming layers (our preliminary checks confirm these obstructions). This strongly suggests that if the base exists it must be highly asymmetric.

References

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